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Matrix Gravity and Massive Colored Gravitons

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Abstract

We formulate a theory of gravity with a matrix-valued complex vierbein based on the $SL(2N,\mathbb{C})\otimes SL(2N,\mathbb{C})$ gauge symmetry. The theory is metric independent, and before symmetry breaking all fields are massless. The symmetry is broken spontaneously and all gravitons corresponding to the broken generators acquire masses. If the symmetry is broken to $SL(2,\mathbb{C})$ then the spectrum would correspond to one massless graviton coupled to $2N^2-1$ massive gravitons. A novel feature is the way the fields corresponding to non-compact generators acquire kinetic energies with correct signs. Equally surprising is the way Yang-Mills gauge fields acquire their correct kinetic energies through the coupling to the non-dynamical antisymmetric components of the vierbeins.

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1 Introduction

The basic interactions of the gravitational field could be easily deduced by promoting the global Lorentz invariance of the Dirac equation to a local one. This simple, yet powerful observation, was first made by Weyl [1] and Cartan [2], and later developed by Utiyama [3] and Kibble [4], to formulate Einstein's theory of gravity as a gauge theory of $SL(2,\mathbb{C})$. The crucial property that guarantees masslessness of the graviton is invariance under diffeomorphisms. Dynamical theories of gauge fields are based on internal symmetries. The space-time symmetry group $SL(2,\mathbb{C})$ and internal symmetry groups are distinct and not unified. According to the Coleman-Mandula theorem [5] the only possible extensions to space-time symmetries are internal ones. There is, however, the exception of supersymmetries which are based on extending the Poincare symmetry by grading the algebra.

Some time ago, Isham, Salam and Strathdee [6] proposed a marriage of $SL(2,\mathbb{C})$ and $SU(3)$ symmetries to describe the interaction of massive spin 2^+ nonets by gauging the unified $SL(6,\mathbb{C})$ group. Their construction is analogous to the first order formalism of gravity where a matrix-valued connection vector is introduced as a conjugate variable to the matrix-valued gauge field. This method insures that degrees of freedom associated with the non-compact components of $SL(6,\mathbb{C})$ do not propagate. This theory was intended to describe strong interactions and the space-time metric was taken to be flat Minkowski.

To unify gravity with internal symmetries, the mechanism introduced in [6] can serve as a starting point, with the graviton promoted to become matrix-valued. To insure diffeomorphism invariance of matrix-valued expressions, no metric is introduced from outside and the action is required to be a four-form on a four-dimensional manifold. The natural group to consider that marries the $SL(2,\mathbb{C})$ symmetry and the internal $U(N)$ symmetry is $SL(2N,\mathbb{C})$. The $8N^2-2$ generators of $SL(2N,\mathbb{C})$ are formed from the tensor products of the generators of $SL(2,\mathbb{C})$ and generators of $U(N)$. An attempt to formulate matrix gravity based on the metric approach tensored to $U(N)$ matrices was considered by Avramidi [7]. However, in [7] all gravitons have unbroken $U(N)$ symmetry and are all massless which is non-physical. More recently he obtained a unique gravity action based on the spectral expansion of non-Laplace type operator [8]. That formulation differs from the approach considered here as we take the mixing between the $SL(2,\mathbb{C})$ space-time symmetry and the $U(N)$ internal symmetry to be non-trivial, and

where after symmetry breaking, all gravitons except one become massive. As a starting point, the vierbein can be taken as a one form transforming under the vector representation of $SL(2N,\mathbb{C})$ symmetry. In the Dirac basis the vierbein will have two components, a vector and pseudo-vector. When expressed in terms of components, this corresponds to a complex matrix-valued vierbein. It turns out that finding a Lagrangian with physically acceptable propagators for all components of the complex matrix-valued vierbein, is not possible within this setting. The reason is that the vector and pseudo-vector components of the vierbein will get kinetic energies of opposite signs. In analogy with what was done in reference [6], the correct strategy to adopt is to extend the gauge group to $SL(2N,\mathbb{C})\otimes SL(2N,\mathbb{C})$. This will remedy the problem of getting the correct signs for the kinetic energies. It will allow us to impose torsion constraints which could then be used to solve for the spin-connection parts of the gauge fields in terms of the vierbeins. It will also be necessary to introduce scalar fields that spontaneously break the symmetry $SL(2N,\mathbb{C})\otimes SL(2N,\mathbb{C})$ to $SL(2,\mathbb{C})$ [6]. This will then permit the coupling of the gravitational sector to matter. Diffeomorphism invariance can only protect one metric, a scalar under the $SU(N)$ factors, from getting a mass. The massive gravitons, except for one singlet, are in a representation of $SU(N)\times SU(N)$ and are therefore colored. Here we shall not worry about the consistency of massive spin-2 theories [9], [10], [11], as this problem has been recently resolved as shown in [12] and [13].The plan of this paper is as follows. In section 2 the gauge fields with their transformations and curvatures are introduced, and torsion free conditions are imposed. In section 3 a metric independent gauge invariant Lagrangian is constructed, and Higgs fields necessary for breaking the symmetry spontaneously are used. In section 4 the spectrum of the Lagrangian is analyzed by deriving the quadratic terms perturbatively. Section 5 is the conclusion.

2 Matrix Gravity

As a starting point we introduce the group $SL(2N,\mathbb{C})\otimes SL(2N,\mathbb{C})$ defined as the complex extension of $SL(2N,\mathbb{C})$ with complex gauge parameters Ω.

This has the following representation in the Dirac basis[2]:

$$\Omega = \exp\omega,$$
$$\omega = P_+\left(\omega_l + \frac{i}{4}l^{ab}\sigma_{ab}\right) + P_-\left(\omega_r + \frac{i}{4}r^{ab}\sigma_{ab}\right),$$

where, $\det\Omega = 1$, and

$$\omega_l = \omega_1 + i\omega_2,$$
$$\omega_r = \omega_1^{'} + i\omega_2^{'},$$
$$P_\pm = \frac{1}{2}(1 \pm \gamma_5).$$

The gauge parameters ω_1, ω_2, $\omega_1^{'}$, $\omega_2^{'}$, l^{ab}, and r^{ab} are Hermitian $N \times N$ matrices. The gauge fields are one forms

$$A = A_\mu dx^\mu,$$

where A_μ are defined by

$$A_\mu = P_+\left(a_\mu + \frac{i}{4}B_\mu^{ab}\sigma_{ab}\right) + P_-\left(b_\mu + \frac{i}{4}C_\mu^{ab}\sigma_{ab}\right),$$

and satisfy the conditions $Tr\,(P_\pm A_\mu) = 0$. The $SU(N) \times SU(N)$ gauge fields a_μ and b_μ are complex

$$a_\mu = a_{1\mu} + ia_{2\mu},$$
$$b_\mu = b_{1\mu} + ib_{2\mu},$$

and the component gauge fields $a_{1\mu}$, $a_{2\mu}$, $b_{1\mu}$, $b_{2\mu}$, B_μ^{ab} and C_μ^{ab} are taken to be Hermitian $N \times N$ matrices subject to the conditions

$$Tr\,(a_\mu) = 0 = Tr\,(b_\mu).$$

Under a gauge transformation, the gauge fields transform as

$$A_\mu \rightarrow \Omega A_\mu \Omega^{-1} + \Omega\partial_\mu\Omega^{-1}.$$

[2] We use the same notation and set of Dirac gamma matrices as reference [14] except for γ_5 which is replaced here with $-i\gamma_5$.

The complex conjugate of the group element Ω is defined by

$$\overline{\Omega} = \gamma_0 \Omega^{\dagger} \gamma_0 = \exp \overline{\omega},$$

so that

$$\overline{\omega} = P_{+} \left(\overline{\omega_r} - \frac{i}{4} r^{ab} \sigma_{ab} \right) + P_{-} \left(\overline{\omega_l} - \frac{i}{4} l^{ab} \sigma_{ab} \right),$$

where

$$\overline{\omega_l} = \omega_1 - i\omega_2, \qquad \overline{\omega_r} = \omega_1^{'} - i\omega_2^{'}.$$

The complex conjugate gauge fields are then

$$\overline{A}_{\mu} = P_{+} \left(\overline{b}_{\mu} - \frac{i}{4} C_{\mu}^{ab} \sigma_{ab} \right) + P_{-} \left(\overline{a}_{\mu} - \frac{i}{4} B_{\mu}^{ab} \sigma_{ab} \right),$$

where

$$\overline{a}_{\mu} = a_{1\mu} - i a_{2\mu}, \qquad \overline{b}_{\mu} = b_{1\mu} - i b_{2\mu},$$

transform as

$$\overline{A}_{\mu} \rightarrow \overline{\Omega}^{-1} \overline{A}_{\mu} \overline{\Omega} - \overline{\Omega}^{-1} \partial_{\mu} \overline{\Omega}.$$

Next, we introduce the two matrix valued vierbeins one forms

$$L = L_{\mu} dx^{\mu}, \qquad L^{'} = L_{\mu}^{'} dx^{\mu},$$

where

$$L_{\mu} = \left(P_{+} E_{\mu}^{a} + P_{-} F_{\mu}^{a} \right) \gamma_a,$$
$$L_{\mu}^{'} = \left(P_{+} F_{\mu}^{'a} + P_{-} E_{\mu}^{'a} \right) \gamma_a,$$

and these transform under product representations of the two groups $SL(2N, \mathbb{C})$:

$$L_{\mu} \rightarrow \Omega L_{\mu} \overline{\Omega}$$
$$L_{\mu}^{'} \rightarrow \overline{\Omega}^{-1} L_{\mu}^{'} \Omega^{-1}$$

Notice that

$$\overline{L}_{\mu} = \gamma_0 L_{\mu}^{\dagger} \gamma_0 = L_{\mu}$$

and

$$\overline{L_{\mu}^{'}} = \gamma_0 L_{\mu}^{'\dagger} \gamma_0 = L_{\mu}^{'}$$

and the combinations $L_\mu L_\nu^{'}$ and $L_\mu^{'} L_\nu$ transform as

$$L_\mu L_\nu^{'} \to \Omega L_\mu L_\nu^{'} \Omega^{-1},$$
$$L_\mu^{'} L_\nu \to \overline{\Omega}^{-1} L_\mu^{'} L_\nu \overline{\Omega},$$

It is instructive to write down the transformations for the component fields E_μ^a, $E_\mu^{'a}$, F_μ^a and $F_\mu^{'a}$,

$$\begin{aligned}
\delta E_\mu^a &= \left\{\alpha_1, E_\mu^a\right\} + i\left[\alpha_2, E_\mu^a\right] - \frac{1}{2}\left\{l^{ab}, E_{\mu b}\right\} - \frac{i}{2}\left[\widetilde{l}^{ab}, E_{\mu b}\right], \\
\delta F_\mu^a &= \left\{\beta_1, F_\mu^a\right\} + i\left[\beta_2, F_\mu^a\right] - \frac{1}{2}\left\{r^{ab}, F_{\mu b}\right\} + \frac{i}{2}\left[\widetilde{r}^{ab}, F_{\mu b}\right], \\
\delta E_\mu^{'a} &= \left\{\alpha_1, E_\mu^{'a}\right\} + i\left[\alpha_2, E_\mu^{'a}\right] - \frac{1}{2}\left\{l^{ab}, E_{\mu b}^{'}\right\} + \frac{i}{2}\left[\widetilde{l}^{ab}, E_{\mu b}^{'}\right], \\
\delta F_\mu^{'a} &= \left\{\beta_1, F_\mu^{'a}\right\} + i\left[\beta_2, F_\mu^{'a}\right] - \frac{1}{2}\left\{r^{ab}, F_{\mu b}^{'}\right\} - \frac{i}{2}\left[\widetilde{r}^{ab}, F_{\mu b}^{'}\right],
\end{aligned}$$

where

$$\widetilde{l}^{ab} = \frac{1}{2}\epsilon^{abcd} l_{cd}, \qquad \widetilde{r}^{ab} = \frac{1}{2}\epsilon^{abcd} r_{cd}.$$

It is clear that one can only identify the fields $E_\mu^{'a} = E_\mu^a$ and $F_\mu^{'a} = F_\mu^a$ in the special case of $SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C})$ [15] as then, the last terms in the previous transformations, which occur with opposite signs vanish. It was shown in reference [15] that, in this $N = 1$ case, it is not essential to consider $SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C})$ and it is sufficient to have the group $SL(2,\mathbb{C})$ only. It is instructive, however, to work out the details in this case to illustrate the generation of mass through the Higgs mechanism.

To avoid doubling the dynamical degrees of freedom, we shall impose constraints on the system so that only one pair of E_μ^a and F_μ^a propagate.

The curvature of the one-forms A and $\overline{A}$ are defined by

$$F = dA + A^2 = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu,$$
$$\overline{F} = d\overline{A} - \overline{A}^2 = \frac{1}{2}\overline{F}_{\mu\nu} dx^\mu \wedge dx^\nu,$$

which give the following field strengths

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right],$$
$$\overline{F}_{\mu\nu} = \partial_\mu \overline{A}_\nu - \partial_\nu \overline{A}_\mu - \left[\overline{A}_\mu, \overline{A}_\nu\right].$$

These transform as

$$
\begin{aligned}
F_{\mu\nu} &\rightarrow \Omega F_{\mu\nu}\Omega^{-1}, \\
\overline{F}_{\mu\nu} &\rightarrow \overline{\Omega}^{-1}\overline{F}_{\mu\nu}\overline{\Omega}.
\end{aligned}
$$

Next we introduce the analogue of torsion forms

$$
\begin{aligned}
T &= dL + AL + LA^{'} = \frac{1}{2}T_{\mu\nu}dx^{\mu} \wedge dx^{\nu}, \\
T^{'} &= dL^{'} + AL^{'} + L^{'}A^{'} = \frac{1}{2}T^{'}_{\mu\nu}dx^{\mu} \wedge dx^{\nu}.
\end{aligned}
$$

The tensors $T_{\mu\nu}$ and $T^{'}_{\mu\nu}$ can be expanded in the Dirac basis

$$
\begin{aligned}
T_{\mu\nu} &= \left(P_{+}T^{a}_{\mu\nu}\left(E\right) + P_{-}T^{a}_{\mu\nu}\left(F\right)\right)\gamma_{a}, \\
T^{'}_{\mu\nu} &= \left(P_{+}T^{a}_{\mu\nu}\left(F^{'}\right) + P_{-}T^{a}_{\mu\nu}\left(E^{'}\right)\right)\gamma_{a},
\end{aligned}
$$

where

$$
\begin{aligned}
T^{a}_{\mu\nu}\left(E\right) &= \nabla_{\mu}E^{a}_{\nu} - \nabla_{\nu}E^{a}_{\mu}, \\
T^{a}_{\mu\nu}\left(F\right) &= \nabla_{\mu}F^{a}_{\nu} - \nabla_{\nu}F^{a}_{\mu},
\end{aligned}
$$

and similarly for $T^{a}_{\mu\nu}\left(E^{'}\right)$ and $T^{a}_{\mu\nu}\left(F^{'}\right)$. The covariant derivatives are given by

$$
\begin{aligned}
\nabla_{\mu}E^{a}_{\nu} &= \partial_{\mu}E^{a}_{\nu} + \left\{a^{1}_{\mu}, E^{a}_{\nu}\right\} + i\left[a^{2}_{\mu}, E^{a}_{\nu}\right] + \frac{1}{2}\left\{B^{ab}_{\mu}, E_{\nu b}\right\} + \frac{i}{2}\left[\widetilde{B}^{ab}_{\mu}, E_{\nu b}\right], \\
\nabla_{\mu}E^{'a}_{\nu} &= \partial_{\mu}E^{'a}_{\nu} + \left\{a^{1}_{\mu}, E^{'a}_{\nu}\right\} + i\left[a^{2}_{\mu}, E^{'a}_{\nu}\right] + \frac{1}{2}\left\{B^{ab}_{\mu}, E^{'}_{\nu b}\right\} - \frac{i}{2}\left[\widetilde{B}^{ab}_{\mu}, E^{'}_{\nu b}\right], \\
\nabla_{\mu}F^{a}_{\nu} &= \partial_{\mu}F^{a}_{\nu} + \left\{b^{1}_{\mu}, F^{a}_{\nu}\right\} + i\left[b^{2}_{\mu}, F^{a}_{\nu}\right] + \frac{1}{2}\left\{C^{ab}_{\mu}, F_{\nu b}\right\} - \frac{i}{2}\left[\widetilde{C}^{ab}_{\mu}, F_{\nu b}\right], \\
\nabla_{\mu}F^{'a}_{\nu} &= \partial_{\mu}F^{'a}_{\nu} + \left\{b^{1}_{\mu}, F^{'a}_{\nu}\right\} + i\left[b^{2}_{\mu}, F^{'a}_{\nu}\right] + \frac{1}{2}\left\{C^{ab}_{\mu}, F^{'}_{\nu b}\right\} + \frac{i}{2}\left[\widetilde{C}^{ab}_{\mu}, F^{'}_{\nu b}\right],
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
\widetilde{B}^{ab}_{\mu} &= \frac{1}{2}\epsilon^{abcd}B_{\mu cd}, \\
\widetilde{C}^{ab}_{\mu} &= \frac{1}{2}\epsilon^{abcd}C_{\mu cd}.
\end{aligned}
$$

From the transformation properties of E_μ^a, F_μ^a, $E_\mu^{'a}$ and $F_\mu^{'a}$ it is clear that it is not possible to make the identification

$$
\begin{aligned}
E_\mu^{'a} &= E_\mu^a, \\
F_\mu^{'a} &= F_\mu^a,
\end{aligned}
$$

without breaking the $SL(2N,\mathbb{C})\otimes SL(2N,\mathbb{C})$ to $SL(2,\mathbb{C})$. In analogy with the Cartan formulation of gravity, we shall impose the following zero torsion constraints

$$
\begin{aligned}
T_{\mu\nu} &= 0, \\
T_{\mu\nu}^{'} &= 0.
\end{aligned}
$$

The number of independent components in $T_{\mu\nu}^a(E)$, $T_{\mu\nu}^a(F)$ or $T_{\mu\nu}^a\left(E^{'}\right)$ and $T_{\mu\nu}^a\left(F^{'}\right)$, match the number of independent components in B_μ^{ab} and C_μ^{ab}. We can then use the $T_{\mu\nu}=0$ constraints to solve for B_μ^{ab} in terms of the fields a_μ, E_μ^a and for C_μ^{ab} in terms of b_μ, F_μ^a. Alternatively we can use the constraint $T_{\mu\nu}^{'}=0$ to solve for B_μ^{ab} in terms of the fields a_μ, $E_\mu^{'a}$ and for C_μ^{ab} in terms of b_μ, $F_\mu^{'a}$. Therefore, equating the two sets of solutions give a complicated relation between a_μ, E_μ^a, $E_\mu^{'a}$, as well as another relation between b_μ, F_μ^a and $F_\mu^{'a}$. These relations being matrix equations could only be solved perturbatively. The important point is that the constraints reduce the number of independent fields from two sets of complex matrix-valued vierbeins to one set. After writing the action, we shall analyze the dynamical degrees of freedom.

The components of the curvature $F_{\mu\nu}$ are given by

$$
F_{\mu\nu} = P_+\left(a_{\mu\nu}+\frac{i}{4}B_{\mu\nu}^{ab}\sigma_{ab}\right)+P_-\left(b_{\mu\nu}+\frac{i}{4}C_{\mu\nu}^{ab}\sigma_{ab}\right),
$$

where

$$
\begin{aligned}
a_{\mu\nu} &= \partial_\mu a_\nu-\partial_\nu a_\mu-\frac{1}{8}\left[B_\mu^{ab},B_{\nu ab}\right]-\frac{1}{16}\epsilon_{abcd}\left[B_\mu^{ab},B_\nu^{cd}\right], \\
b_{\mu\nu} &= \partial_\mu b_\nu-\partial_\nu b_\mu-\frac{1}{8}\left[C_\mu^{ab},C_{\nu ab}\right]-\frac{1}{16}\epsilon_{abcd}\left[C_\mu^{ab},C_\nu^{cd}\right], \\
B_{\mu\nu}^{\ ab} &= \partial_\mu B_\nu^{\ ab}+\frac{1}{2}\left\{B_\mu^{\ ac},B_{\nu c}^{\ \ b}\right\}+i\left[B_\mu^{\ ab},a_\nu^2\right]+i\left[\widetilde{B}_\mu^{\ ab},a_\nu^1\right]-\mu\leftrightarrow\nu, \\
C_{\mu\nu}^{\ ab} &= \partial_\mu C_\nu^{\ ab}-\partial_\nu C_\mu^{\ ab}+\frac{1}{2}\left\{C_\mu^{\ ac},C_{\nu c}^{\ \ b}\right\}+i\left[C_\mu^{\ ab},b_\nu^2\right]+i\left[\widetilde{C}_\mu^{\ ab},b_\nu^1\right]-\mu\leftrightarrow\nu.
\end{aligned}
$$

The complex conjugate gauge field strength $\overline{F}$ are not independent and are given by

$$\overline{F}_{\mu\nu} = P_+ \left(\overline{b}_{\mu\nu} - \frac{i}{4} C^{ab}_{\mu\nu} \sigma_{ab} \right) + P_- \left(\overline{a}_{\mu\nu} - \frac{i}{4} B^{ab}_{\mu\nu} \sigma_{ab} \right).$$

3 Metric Independent Lagrangian

The model we want to construct should also include the Einstein gravitational field. Therefore, we will not use the metric given on the manifold M, but make the requirement that it should arise dynamically from the vierbeins. This will impose the very strong constraint that every term in the Lagrangian should be a four-form, to insure diffeomorphism invariance. With the fields introduced so far, it is possible to build a limited number of gauge invariant terms which are also four-forms. These are

$$\frac{1}{4} \int_M Tr \left(i \left(\alpha + \beta \gamma_5 \right) L L^{'} F + i \left(\overline{\alpha} + \overline{\beta} \gamma_5 \right) L^{'} L \overline{F} + \left(i\lambda + \gamma_5 \eta \right) L L^{'} L L^{'} \right),$$

where $\alpha = \alpha_1 + i\alpha_2$, $\overline{\alpha} = \alpha_1 - i\alpha_2$, $\beta = \beta_1 + i\beta_2$, $\overline{\beta} = \beta_1 - i\beta_2$ are coupling constants.

Before proceeding, it is instructive to consider the simple case of $N = 1$. Here the $SL(2, \mathbb{C})$ conditions imply that $a_\mu = 0 = b_\mu$, and the two conditions $T_{\mu\nu} = T^{'}_{\mu\nu} = 0$ could be solved to give

$$\begin{aligned}
E^{'a}_{\mu} &= E^{a}_{\mu} = e^{a}_{\mu}, \\
F^{'a}_{\mu} &= F^{a}_{\mu} = f^{a}_{\mu} \\
B^{ab}_{\mu} &= \frac{1}{2} e^{\nu a} e^{\rho b} \left(\Omega_{\mu\nu\rho}(e) - \Omega_{\nu\rho\mu}(e) + \Omega_{\rho\mu\nu}(e) \right) \equiv \omega^{ab}_{\mu}(e), \\
C^{ab}_{\mu} &= \frac{1}{2} f^{\nu a} f^{\rho b} \left(\Omega_{\mu\nu\rho}(f) - \Omega_{\nu\rho\mu}(f) + \Omega_{\rho\mu\nu}(f) \right) \equiv \omega^{ab}_{\mu}(f), \\
\Omega_{\mu\nu\rho}(e) &= \left(\partial_\mu e^{c}_{\nu} - \partial_\nu e^{c}_{\mu} \right) e_{\rho c}, \\
\Omega_{\mu\nu\rho}(f) &= \left(\partial_\mu f^{c}_{\nu} - \partial_\nu f^{c}_{\mu} \right) f_{\rho c}.
\end{aligned}$$

In this special case it is possible to identify the fields appearing in L and $L^{'}$ as was shown in [15], because it becomes possible to define an operation of

charge conjugation that transforms L to $L^{'}$. With these simplifications the action reduce to

$$
\begin{aligned}
-\frac{1}{2}\int\limits_{M} d^4x\epsilon^{\mu\nu\kappa\lambda}\Bigg(\Bigg(\left(\alpha_2-\beta_1\right)e_{\mu a}e_{\nu b}+\frac{1}{2}\left(\alpha_1+\beta_2\right)\epsilon_{abcd}e_\mu^c e_\nu^d\Bigg)B_{\kappa\lambda}^{ab} \\
+\left(\left(\alpha_2+\beta_1\right)f_{\mu a}f_{\nu b}-\frac{1}{2}\left(\alpha_1-\beta_2\right)\epsilon_{abcd}f_\mu^c f_\nu^d\right)C_{\kappa\lambda}^{ab} \\
+\epsilon_{abcd}\left(\left(\lambda-\eta\right)e_\mu^a e_\nu^b e_\kappa^c e_\lambda^d+\left(\lambda+\eta\right)f_\mu^a f_\nu^b f_\kappa^c f_\lambda^d\right)\Bigg).
\end{aligned}
$$

It is clear that this action describes two non-interacting vierbeins, and what is needed is to add to the action mixing terms to give mass to one of the metrics. This, however, is not possible without breaking the symmetry spontaneously from $SL(2,\mathbb{C})\otimes SL(2,\mathbb{C})$ to $SL(2,\mathbb{C})$. Such program was carried many years ago in [17] for strong supergravity based on spontaneously breaking the graded orthosymplectic gauge symmetry $OSP(2,2;1)\times OSP(2,2;1)$ to $SL(2,\mathbb{C})$. We shall return to this simple example of $N=1$ when we analyze the spectrum.

Introduce the Higgs fields H and $H^{'}$ which transform, respectively, as L and $L^{'}$:

$$
\begin{aligned}
H &\rightarrow \Omega H\overline{\Omega}, \\
H^{'} &\rightarrow \overline{\Omega}^{-1}H^{'}\Omega^{-1},
\end{aligned}
$$

and subject to the reality conditions $H=\overline{H}$, $H^{'}=\overline{H^{'}}$. The component forms of H and $H^{'}$are given by

$$
\begin{aligned}
H &= \left(h_1+\gamma_5 h_2+\frac{1}{4}h^{ab}\sigma_{ab}\right), \\
H^{'} &= \left(h_1^{'}+\gamma_5 h_2^{'}+\frac{1}{4}h^{'ab}\sigma_{ab}\right),
\end{aligned}
$$

where all the components are real. However, in performing calculations it proves easier to decompose H and $H^{'}$ in terms of their chiral components

$$
\begin{aligned}
H &= P_+\left(h+\frac{1}{4}h_+^{ab}\sigma_{ab}\right)+P_+\left(h^*+\frac{1}{4}h_-^{ab}\sigma_{ab}\right), \\
H &= P_+\left(h^{'}+\frac{1}{4}h_+^{'ab}\sigma_{ab}\right)+P_+\left(h^{'*}+\frac{1}{4}h_-^{'ab}\sigma_{ab}\right),
\end{aligned}
$$

where $h_{ab} = h_{+ab} + h_{-ab}$ the sum of the self-dual and anti self-dual parts:

$$h_{\pm ab} = \pm \frac{i}{2} \epsilon_{abcd} h_{\pm}^{cd}, \qquad h = h_1 + ih_2,$$
$$h'_{\pm ab} = \pm \frac{i}{2} \epsilon_{abcd} h_{\pm}^{'cd}, \qquad h = h_1 + ih_2.$$

The fact that the scalar field H transforms like L and H' transforms like L' allows us to form new combinations. The choice of terms is limited because of the constraint that every term should be a four-form. The simplest allowed combinations are

$$\frac{1}{4} \int_M Tr \left((i\tau + \gamma_5 \xi)\, LH' LH' LL' + (i\rho + \gamma_5 \chi)\, HL' HL' LL' \right).$$

To the above, we can add kinetic terms to the field H in the form

$$\int_M Tr \left((a + b\gamma_5) \left(\nabla H \nabla H' LL' + \nabla H' \nabla H L' L \right) \right),$$

where

$$\nabla H = dH + AH + H\overline{A},$$
$$\nabla H' = dH' - AH' - H'\overline{A}.$$

Alternatively, it is possible to use the method of non-linear realization [18] to constrain the Higgs fields by gauge invariant conditions of the form

$$Tr \left(P_{\pm} \left(HH' \right)^n \right) = c_{\pm n}, \qquad n = 1, 2, \cdots,$$
$$Tr \left(P_{\pm} H \nabla_{\mu} H' \right) = 0,$$

where c_n are constants. In this way there will be no need to add a potential for the Higgs fields, but instead solve the constraints to eliminate all degrees of freedom. The number of independent constraints is given by the dimension of the coset space $\frac{SL(2N,\mathbb{C}) \otimes SL(2N,\mathbb{C})}{SL(2,\mathbb{C})}$. We shall assume that we can use the constraints to eliminate all degrees of freedom in H and H', by using the gauge freedom to set h_{ab} or h'_{ab} to zero. To see this we write the infinitesimal gauge transformations for H

$$\delta H = \omega H + H\overline{\omega}$$

and then evaluate it in component form, to obtain

$$\begin{aligned}
\delta h &= \omega_l h + h\overline{\omega}_r + \frac{i}{4} l^{ab} h_{+ab} - \frac{i}{4} h_{+ab} r^{ab}, \\
\delta h_{+ab} &= \omega_l h_{+ab} + h_{+ab}\overline{\omega}_r + i l_+^{ab} h - i h r_{+ab} \\
&\quad - \frac{1}{2}\left(h_{+ac} r^{c}_{+b} - h_{+bc} r^{c}_{+a}\right) + \frac{1}{2}\left(l_{+ac} h^{c}_{+b} - l_{+bc} h^{c}_{+a}\right),
\end{aligned}$$

where $l_{\pm ab} = \frac{1}{2}\left(l_{ab} \pm i\widetilde{l}_{ab}\right)$ and $r_{\pm ab} = \frac{1}{2}\left(r_{ab} \pm i\widetilde{r}_{ab}\right)$. We can also write a similar relation for $h^{'}$ and $h^{'}_{+ab}$. By fixing the gauge degrees of freedom it is possible to set

$$\langle H \rangle = h_i \lambda^i, \quad \left\langle H^{'} \right\rangle = h^{'}_i \lambda^i,$$

where h_i and $h^{'}_i$ are constants and λ^i are the Gell-Mann $U(N)$ matrices. Obviously in this unitary gauge the action simplifies, and the mixing terms for the matrix-valued vierbeins give masses to all the gravitons, save one singlet, which is protected to stay massless by diffeomorphism invariance. In the special case where $\langle H \rangle$ and $\left\langle H^{'} \right\rangle$ are diagonal, the symmetry is broken down to $\left(SL(2,\mathbb{C}) \times U(1)\right)^N$. Physically this is not desirable because it will imply the existence of N massless gravitons [19]. To have a realistic theory we have to assume that the symmetry is broken down to a $SL(2,\mathbb{C})$ so that only one massless graviton remains. In what follows we shall study the spectrum and show that all gravitons acquire Fierz-Pauli mass terms [16] through the Higgs mechanism.

4 The Spectrum

To analyze the spectrum, we first derive the component form of the action. The full action, when expressed in terms of components, is complicated and is given in the appendix. In what follows we shall determine the spectrum by studying perturbatively the quadratic part of the action.

To proceed, it is important to solve the torsion constraints and determine the independent degrees of freedom. Because of the matrix nature of the equations, these could only be solved perturbatively. To simplify matters we use the basis of Gell-Mann matrices λ^i for $U(N)$ satisfying the commutation

and anticommutation relations

$$\left[\lambda^i, \lambda^j\right] = 2if^{ijk}\lambda^k,$$
$$\left\{\lambda^i, \lambda^j\right\} = 2d^{ijk}\lambda^k,$$

and normalized by the condition $Tr\left(\lambda^i\lambda^j\right) = 2\delta^{ij}$. It is also important to distinguish the diagonal $U(1)$ matrix $\lambda^0 = \sqrt{\frac{2}{N}}1_N$ from the $SU(N)$ matrices λ^I, $I = 1, \cdots, N^2-1$ where $i = 0, I$. Using the relations

$$f^{0IJ} = 0, \qquad d^{000} = \sqrt{\frac{2}{N}},$$
$$d^{0IJ} = d^{I0J} = d^{IJ0} = \sqrt{\frac{2}{N}}\delta^{IJ},$$

we can decompose the torsion equations for E_μ^{ai} into the form:

$$\partial_\mu E_\nu^{ai} + 2d^{ijk}a_\mu^{1j}E_\nu^{ak} + 2f^{ijk}a_\mu^{2j}E_\nu^{ak} + d^{ijk}B_\mu^{abj}E_{\nu b}^k + f^{ijk}\widetilde{B}_\mu^{abj}E_{\nu b}^k - \mu \leftrightarrow \nu = 0.$$

A similar equation holds for $E_\mu^{\prime ai}$ where the only difference is an opposite sign for the term with $\widetilde{B}_\mu^{abj}$. The above condition can be decomposed into two sets

$$\sqrt{\frac{N}{2}}\partial_\mu E_\nu^{a0} + 2a_\mu^{1I}E_\nu^{aI} + B_\mu^{ab0}E_{\nu b}^0 + B_\mu^{abI}E_{\nu b}^I - \mu \leftrightarrow \nu = 0,$$

$$\partial_\mu E_\nu^{aI} + 2\sqrt{\frac{2}{N}}a_\mu^{1I}E_\nu^{a0} + 2d^{IJK}a_\mu^{1J}E_\nu^{aK} + 2f^{IJK}a_\mu^{2J}E_\nu^{aK}$$
$$+\sqrt{\frac{2}{N}}B_\mu^{abI}E_{\nu b}^0 + d^{IJK}B_\mu^{abJ}E_{\nu b}^K + f^{IJK}\widetilde{B}_\mu^{abJ}E_{\nu b}^K - \mu \leftrightarrow \nu = 0.$$

To solve these equations perturbatively we consider fluctuations around a flat background. We therefore write

$$E_\mu^{a0} = \sqrt{\frac{N}{2}}\delta_\mu^a + E_\mu^{a0(1)} + E_\mu^{a0(2)} + \cdots,$$
$$E_\mu^{aI} = E_\mu^{aI(1)} + E_\mu^{aI(2)} + \cdots,$$
$$B_\mu^{ab0} = B_\mu^{ab0(0)} + B_\mu^{ab0(1)} + B_\mu^{ab0(2)} + \cdots,$$
$$B_\mu^{abI} = B_\mu^{abI(1)} + B_\mu^{abI(2)} + \cdots,$$
$$a_\mu^I = a_\mu^{I(1)} + a_\mu^{I(2)} + \cdots.$$

These expansions, when substituted in the above torsion constraints, could be solved perturbatively for B_μ^{ab0} and B_μ^{abI} in terms of E_μ^{a0}, E_μ^{aI} and a_μ^I. It is possible to absorb the corrections $E_\mu^{a0(1)}$, $E_\mu^{a0(2)}, \cdots$, by using an arbitrary curved background e_μ^a. Similarly we can absorb the corrections to a_μ^I by redefining it. We therefore write

$$\begin{aligned} E_\mu^{a0} &= \sqrt{\frac{N}{2}} e_\mu^a, \\ a_\mu^I &= a_\mu^{I1} + i a_\mu^{I2}, \end{aligned}$$

and consider these terms to be of order zero. We then have, to zeroth order

$$\partial_\mu e_\nu^a + \sqrt{\frac{2}{N}} B_\mu^{ab0(0)} e_{\nu b} - \mu \leftrightarrow \nu = 0,$$

which could be easily solved to give

$$B_\mu^{ab0(0)} = \sqrt{\frac{N}{2}} \omega_\mu^{ab}\left(e\right),$$

where $\omega_\mu^{ab}\left(e\right)$ is the usual spin-connection of the vierbein e_μ^a .

To first order we then have

$$\begin{aligned} B_\mu^{ab0(1)} e_{\nu b} - \mu \leftrightarrow \nu = 0, \\ \partial_\mu E_\nu^{aI(1)} + 2 a_\mu^{1I} e_\nu^a + B_\mu^{abI(1)} e_{\nu b} + \omega_\mu^{ab}\left(e\right) E_{\nu b}^{I(1)} - \mu \leftrightarrow \nu = 0, \end{aligned}$$

which give the solutions

$$\begin{aligned} B_\mu^{ab0(1)} &= 0, \\ B_\mu^{abI(1)} &= \frac{1}{2}\left(\Omega_{\mu\nu\rho}^{I(1)} - \Omega_{\nu\rho\mu}^{I(1)} + \Omega_{\rho\mu\nu}^{I(1)}\right) e^{a\nu} e^{b\rho}, \end{aligned}$$

where

$$\Omega_{\mu\nu\rho}^{I(1)} = D_\mu^{(g)} E_\nu^{aI(1)} e_{a\rho} + 4 a_\mu^{1I} g_{\nu\rho} - \mu \leftrightarrow \nu.$$

The covariant derivative D_μ is taken with respect to the background metric $g_{\mu\nu} = e_\mu^a e_{\nu a}$. This process can be continued to second order to give the constraints

$$2 a_\mu^{1I} E_\nu^{aI(1)} + B_\mu^{ab0(2)} e_{\nu b} + B_\mu^{abI(1)} E_{\nu b}^{I(1)} - \mu \leftrightarrow \nu = 0,$$

$$\partial_\mu E_\nu^{aI(2)} + 2d^{IJK} a_\mu^{1J} E_\nu^{aK(1)} + 2f^{IJK} a_\mu^{2J} E_\nu^{aK(1)} + B_\mu^{abI(2)} e_{\nu b}$$
$$+\omega_\mu^{ab}(e) E_{\nu b}^{I(2)} + d^{IJK} B_\mu^{abJ(1)} E_{\nu b}^{K(1)} + f^{IJK} \widetilde{B}_\mu^{abJ(1)} E_{\nu b}^{K(1)} - \mu \leftrightarrow \nu = 0.$$

The solution of these equations are

$$B_\mu^{ab0(2)} = \frac{1}{2}\left(\Omega_{\mu\nu\rho}^{0(2)} - \Omega_{\nu\rho\mu}^{0(2)} + \Omega_{\rho\mu\nu}^{0(2)}\right) e^{a\nu} e^{b\rho},$$
$$B_\mu^{abI(2)} = \frac{1}{2}\left(\Omega_{\mu\nu\rho}^{I(2)} - \Omega_{\nu\rho\mu}^{I(2)} + \Omega_{\rho\mu\nu}^{I(2)}\right) e^{a\nu} e^{b\rho},$$

where

$$\Omega_{\mu\nu\rho}^{0(2)} = \left(2a_\mu^{1I} E_\nu^{aI(1)} + B_\mu^{abI(1)} E_{\nu b}^{I(1)}\right) e_{\rho a} - \mu \leftrightarrow \nu,$$
$$\Omega_{\mu\nu\rho}^{I(2)} = \left(D_\mu^{(g)} E_\nu^{aI(2)} + 2a_\mu^{1I} e_\nu^a + 2d^{IJK} a_\mu^{1J} E_\nu^{aK(1)} + 2f^{IJK} a_\mu^{2J} E_\nu^{aK(1)}\right.$$
$$\left. + d^{IJK} B_\mu^{abJ(1)} E_{\nu b}^{K(1)} + f^{IJK} \widetilde{B}_\mu^{abJ(1)} E_{\nu b}^{K(1)} - \mu \leftrightarrow \nu\right).$$

Next, we solve the torsion constraints on $E_\mu^{\prime ai}$, which give B_μ^{abi} in terms of $E_\mu^{\prime ai}$ and a_μ^i. By expanding around the background

$$E_\mu^{\prime a0} = \sqrt{\frac{N}{2}} e_\mu^{\prime a},$$
$$E_\mu^{\prime aI} = E_\mu^{\prime aI(1)} + E_\mu^{\prime aI(2)} + \cdots,$$

and equating perturbatively the two expressions obtained for B_μ^{abi} we deduce that

$$E_\mu^{\prime a0} = \sqrt{\frac{N}{2}} e_\mu^a,$$
$$E_\mu^{\prime aI(1)} = E_\mu^{aI(1)}.$$

However, the relation between $E_\mu^{\prime aI(2)}$ and $E_\mu^{aI(2)}$ is complicated. These are $24(N^2-1)$ constraints on the $40(N^2-1)$ fields $E_\mu^{\prime aI(2)}$, $E_\mu^{aI(2)}$, a_μ^{1I} and a_μ^{2I}. This leaves $16(N^2-1) + 16 = 16N^2$ unconstrained variables corresponding to the components of a matrix-valued vierbein.

A similar analysis can be performed on the fields F_{μ}^{ai} and $F_{\mu}^{'ai}$ to show that the unconstrained variables correspond to a second matrix valued vierbein. To lowest orders, the expansions are

$$
\begin{aligned}
F_{\mu}^{a0} &= \sqrt{\frac{N}{2}} f_{\mu}^{a}, \\
F_{\mu}^{aI} &= F_{\mu}^{aI(1)} + F_{\mu}^{aI(2)} + \cdots , \\
F_{\mu}^{'aI} &= F_{\mu}^{aI(1)} + F_{\mu}^{'aI(2)} + \cdots , \\
F_{\mu}^{'a0} &= f_{\mu}^{a} + F_{\mu}^{a0(1)} + F_{\mu}^{a0(2)} + \cdots , \\
b_{\mu}^{I} &= b_{\mu}^{I1} + i b_{\mu}^{I2},
\end{aligned}
$$

where again the resulting relation between $F_{\mu}^{'aI(2)}$ and $F_{\mu}^{aI(2)}$ is complicated.

To determine the dynamical degrees of freedom, we expand the action to terms of order 2. This will give kinetic and mass terms. All of this could be done in a covariant way by expanding around a curved background determined by the two fields e_{μ}^{a} and f_{μ}^{a}. For simplicity, we group the terms according to their perturbative order in terms of fluctuations. To zeroth order we have

$$
\begin{aligned}
\int\limits_{M} d^4x \epsilon^{\mu\nu\kappa\lambda} \Big(\epsilon_{abcd} \big(& (\lambda - \eta) e_{\mu}^{a} e_{\nu}^{b} e_{\kappa}^{c} e_{\lambda}^{d} + (\lambda + \eta) f_{\mu}^{a} f_{\nu}^{b} f_{\kappa}^{c} f_{\lambda}^{d} \\
& + C_1 h_i h_i e_{\mu}^{a} e_{\nu}^{b} e_{\kappa}^{c} f_{\lambda}^{d} + C_2 h_i h_i e_{\mu}^{a} f_{\nu}^{b} f_{\kappa}^{c} f_{\lambda}^{d} \\
& - \frac{1}{2} (\alpha_1 - \beta_2) e_{\mu}^{a} e_{\nu}^{b} R_{\kappa\lambda}^{cd}(e) + \frac{1}{2} (\beta_1 - \alpha_2) f_{\mu}^{a} f_{\nu}^{b} R_{\kappa\lambda}^{cd}(f) \Big) \\
& + (\beta_1 - \alpha_2) e_{\mu}^{a} e_{\nu}^{b} R_{\kappa\lambda ab}(e) - (\beta_1 + \alpha_2) f_{\mu}^{a} f_{\nu}^{b} R_{\kappa\lambda ab}(f) \big) ,
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= (\rho - \chi) + (\tau - \xi) l^2, \\
C_2 &= -(\rho + \chi) - (\tau + \xi) l^2,
\end{aligned}
$$

and for simplicity, we have assumed that

$$
h_i^{'} = l h_i .
$$

In reality this is full Lagrangian, in the unitary gauge, for the special case of $N = 1$. It is instructive to see the more details in this simple case. First the

gauge transformations of the Higgs fields simplify to

$$\begin{aligned}\delta h &= \frac{i}{4}\left(l^{ab}-r^{ab}\right)h_{+ab},\\ \delta h_{+ab} &= i\left(l_{+ab}-r_{+ab}\right)h+\frac{1}{2}\left(l_{+ac}-r_{+ac}\right)h^{c}_{+b}-\frac{1}{2}\left(l_{+bc}-r_{+bc}\right)h^{c}_{+a}\end{aligned}$$

This proves that one can use the $\left(l^{ab}-r^{ab}\right)$ gauge freedom to set h_{ab} or $h^{'ab}$ to zero. The constraints on the Higgs fields then give

$$\begin{aligned}hh^{'} &= c_1+ic_2,\\ h\partial_\mu h^{'} &= 0\end{aligned}$$

A solution to these equations is to have h_1, h_2, $h_1^{'}$ and $h_2^{'}$ as constants. The $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ Lagrangian has all the desirable properties and gives unique non-ambiguous interactions. This is to be contrasted with the metric theory where the possible interactions that could be written are infinite. The last two terms in the above action are topological. This Lagrangian gives the interaction of one massless graviton and one massive graviton with a Fierz-Pauli mass term [16]. A full analysis of a similar system was performed many years ago and applied to a theory of massive supergravity [17].

We now continue our analysis of the spectrum in the general case. We expand the fields e^a_μ and f^a_μ around a Minkowski. background

$$\begin{aligned}e_{\mu a} &= \eta_{\mu a}+\overline{e}_{\mu a},\\ f_{\mu a} &= \eta_{\mu a}+\overline{f}_{\mu a},\end{aligned}$$

By imposing conditions that the linear fluctuations in $\overline{e}_{\mu a}$ and $\overline{f}_{\mu a}$ vanish we obtain the following relations between the the free parameters λ, η, τ, ξ, ρ and χ

$$\begin{aligned}\lambda &= \frac{2}{N}\left(\chi+\xi l^2\right)h_ih_i,\\ \eta &= \frac{1}{N}\left(\rho+\tau l^2\right)h_ih_i.\end{aligned}$$

One then checks that using the above conditions, the cosmological constant vanishes, and the mass terms simplify to

$$-6\lambda\int\limits_M d^4x\left(\delta^\mu_a\delta^\nu_b-\delta^\mu_b\delta^\nu_a\right)\left(\overline{e}^a_\mu-\overline{f}^a_{\mu a}\right)\left(\overline{e}^b_\nu-\overline{f}^b_\nu\right),$$

which is of the Fierz-Pauli form. This shows that the combination $\left(\overline{e}_{\mu}^{a}+\overline{f}_{\mu a}^{a}\right)$ stays massless while $\left(\overline{e}_{\mu}^{a}-\overline{f}_{\mu a}^{a}\right)$ acquires a mass.

Next we study the dynamical degrees of freedom associated with the vierbeins E_{μ}^{aI} and F_{μ}^{aI}. First the linear terms are given by

$$\frac{2}{N}\int_{M} d^{4}x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}d^{jkI}h_{j}h_{k}\left(C_{1}\left(3\overline{E}_{\mu}^{aI}f_{\nu}^{b}e_{\kappa}^{c}e_{\lambda}^{d}+\overline{F}_{\mu}^{aI}e_{\nu}^{b}e_{\kappa}^{c}e_{\lambda}^{d}\right)\right.$$
$$\left.+C_{2}\left(\overline{E}_{\mu}^{aI}f_{\nu}^{b}f_{\kappa}^{c}f_{\lambda}^{d}+3\overline{F}_{\mu}^{aI}e_{\nu}^{b}f_{\kappa}^{c}f_{\lambda}^{d}\right)\right).$$

To eliminate this term we impose the constraint on the values h_{i} :

$$d^{jkI}h_{j}h_{k}=0.$$

This will help reduce the mass terms for the $\overline{E}_{\mu}^{aI}$ and $\overline{F}_{\mu}^{aI}$ fields to the form

$$\frac{12}{N}\int_{M} d^{4}x\delta_{ab}^{\mu\nu}\left(\left(\lambda-2\eta\right)\overline{E}_{\mu}^{aI}\overline{E}_{\nu}^{bI}+\left(\lambda+2\eta\right)\overline{F}_{\mu}^{aI}\overline{F}_{\nu}^{bI}\right.$$
$$\left.+2\left(\chi+\xi l^{2}\right)h_{j}h_{k}\left(f_{jMP}f_{kNP}-d_{jMP}d_{kNP}\right)\overline{E}_{\mu}^{aI}\overline{F}_{\nu}^{bI}\right).$$

The mass terms are of the Fierz-Pauli type generated by breaking the gauge symmetry spontaneously. For generic values of h_{i} the mass matrix is non-singular, and can be made positive definite. This way all graviton fields E_{μ}^{aI} and F_{μ}^{aI} acquire mass terms. Note that in the special case when the fields H and $H^{'}$ are diagonal then there will be a preserved $U(1)^{N}$ subgroup of $U(N)$ and it is easy to see that the N combinations $E_{\mu}^{aI}+F_{\mu}^{aI}$ corresponding to the diagonal generators remain massless while all other fields become massive. This, of course, is undesirable and will be avoided.

Next, we study the kinetic terms. It is well known that the above Lagrangian give the correct kinetic terms for the fields e_{μ}^{a} and f_{μ}^{a}. We have to determine whether the fields E_{μ}^{aI} and F_{μ}^{aI} obtain the correct kinetic terms as well, and the nature of the complex $SU(N)$ gauge fields a_{μ}^{I} and b_{μ}^{I}. First we

examine the interaction

$$
\begin{aligned}
&\int_M d^4x\epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}tr\left(\left\{E^a_\mu, E^b_\nu\right\}B^{cd}_{\kappa\lambda}\right) \\
&= 4\int_M d^4x\epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}d^{ijk}E^{ai}_\mu E^{bj}_\nu B^{cdk}_{\kappa\lambda} \\
&= \int_M d^4x\epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}\left(2Ne^a_\mu e^b_\nu R^{cd}_{\kappa\lambda}(e) + 8e^a_\mu E^{bI}_\nu B^{cdI}_{\kappa\lambda} + 4E^{aI}_\mu E^{bI}_\nu R^{cd}_{\kappa\lambda}(e)\right).
\end{aligned}
$$

and concentrate on the middle term. Substituting for B^{abI}_μ from the solution of the torsion constraint, we obtain, to linearized order

$$
\begin{aligned}
2B^I_{\mu ab} = {} & \partial_\mu\left(E^I_{ab} - E^I_{ba}\right) - \partial_a\left(E^I_{\mu b} + E^I_{b\mu}\right) \\
& + \partial_b\left(E^I_{\mu a} + E^I_{a\mu}\right) + 8\left(\eta_{\mu a}a^{1I}_b - \eta_{\mu b}a^{1I}_a\right),
\end{aligned}
$$

where indices are raised and lowered with the Minkowski metric η_{ab}. We also decompose $E^I_{\mu a}$ into symmetric and antisymmetric parts

$$
E^I_{\mu a} = S^I_{\mu a} + T^I_{\mu a},
$$

where $S^I_{\mu a} = S^I_{a\mu}$ is symmetric and $T^I_{\mu a} = -T^I_{a\mu}$ is antisymmetric. We also denote $S^I = \eta^{\mu a}S^I_{\mu a}$. The middle term in the above Lagrangian then gives

$$
\begin{aligned}
&4\int_M d^4x\left(\partial_\mu S^I\partial^\mu S^I - 2\partial_\mu S^I\partial_\nu S^{\mu\nu I} + 2\partial_\nu S^{\mu\nu I}\partial^\kappa S^I_{\mu\kappa} - \partial_\mu S^I_{\nu\kappa}\partial^\mu S^{\nu\kappa I}\right) \\
&+ 8\int_M d^4x a^{1I}_\mu\left(\partial^\mu S^I - \partial_\nu S^{\mu\nu I}\right) \\
&+ 4\int_M d^4x\left(\partial_\mu a^{1I}_\nu - \partial_\nu a^{1I}_\mu\right)T^{\mu\nu I}.
\end{aligned}
$$

The kinetic terms for the symmetric tensor $S_{\mu\nu}$ are of the standard form for a spin-2 field [10]. It is remarkable that the antisymmetric part of $E^I_{\mu a}$ does not acquire a kinetic term, but instead couples to the field strength of a^{1I}_μ. Therefore, to lowest order in the perturbative expansion, it could be considered as an auxiliary field and eliminated. There is also the coupling

$$
\int_M d^4x\epsilon^{\mu\nu\kappa\lambda}e^a_\mu E^{bI}_\nu B^{abI}_{\kappa\lambda},
$$

whose quadratic part simplifies to

$$4\int\limits_{M} d^4x T^{I}_{\mu\nu} \left(\epsilon^{\mu\nu\kappa\lambda} \left(\partial_\kappa a^{1I}_\lambda - \partial_\lambda a^{1I}_\kappa\right)\right).$$

We notice that $T^{I}_{\mu\nu}$ also couples to the field strength of a^{2I}_μ as can be seen by expanding the term

$$\int\limits_{M} d^4x \epsilon^{\mu\nu\kappa\lambda} tr \left(\left\{E^a_\mu, E^{'}_{\nu a}\right\} a^2_{\kappa\lambda}\right),$$

whose lowest order contribution is

$$-4\int\limits_{M} d^4x T^{I}_{\mu\nu} \left(\epsilon^{\mu\nu\kappa\lambda} \left(\partial_\kappa a^{2I}_\lambda - \partial_\lambda a^{2I}_\kappa\right)\right).$$

It is then clear that by eliminating the field $T^{I}_{\mu\nu}$ both a^{1I}_μ and a^{2I}_μ would acquire the regular $SU(N)$ Yang-Mills gauge field strengths. A similar analysis would also apply to the antisymmetric part of $F^{I}_{\mu a}$ giving rise to the propagation of b^{1I}_μ and b^{2I}_μ. In reality since the quadratic terms for $E^{I}_{\mu a}$ and $F^{I}_{\mu a}$ mix, and have to be diagonalized, the kinetic energies of a^{1I}_μ, a^{2I}_μ, b^{1I}_μ and b^{2I}_μ will also mix and have to be diagonalized as well. This shows that the fields $\left(e^a_\mu - f^a_\mu\right)$, $E^{I}_{\mu a}$, $F^{I}_{\mu a}, a^{1I}_\mu$, a^{2I}_μ, b^{1I}_μ and b^{2I}_μ all have correct kinetic energies and all obtain Fierz-Pauli mass terms after the symmetry is broken spontaneously. The combination $\left(e^a_\mu + f^a_\mu\right)$ remains massless and correspond to the usual graviton. It is important to stress that the model is based on a first order Lagrangian, where the gauge spin-connections are determined from the zero torsion conditions to depend on the vierbeins and their derivatives as well as on the Yang-Mills fields a_μ and b_μ. The Yang-Mills fields a_μ and b_μ will only have first order derivatives from the gauge field strengths, but because of the torsion constraints couple to the antisymmetric parts of the vierbeins, which then gives them kinetic energies. In the full non-linear theory, their will be also higher order interactions for the antisymmetric parts of the vierbeins and therefore could not be eliminated as auxiliary fields. The Yang-Mills fields will not have the canonical form for their kinetic energies, but because of gauge invariance, one would expect these terms to have the correct dynamics.

5 Conclusions

We have shown that it is possible to construct a sensible theory of a complex matrix-valued graviton based on the gauge theory of $SL(2N,\mathbb{C})\otimes SL(2N,\mathbb{C})$. Matrix-valued complex vierbeins are introduced, and in analogy with the first order formalism of Einstein gravity, the gauge fields are restricted by imposing torsion like constraints on the vierbeins. The symmetry is broken down spontaneously to $SL(2,\mathbb{C})$ by using two Higgs fields. Constraints are imposed on the Higgs fields, thus breaking the symmetry non-linearly. In the unitary gauge the Higgs fields could be set to constants, thus generating masses to the gravitons. Remarkably, only the symmetric parts of the $2\left(N^2-1\right)$ fields $E_{\mu a}^I$, $F_{\mu a}^I$ acquire kinetic energies, while the antisymmetric parts do not. These, instead, couple to the complex $SU(N)\times SU(N)$ gauge fields a_μ^I, b_μ^I and after being eliminated as auxiliary fields, give them kinetic energies. In addition there is the massless graviton $\left(e_\mu^a+f_\mu^a\right)$ and a massive graviton $\left(e_\mu^a-f_\mu^a\right)$. Since the masses are obtained through the Higgs mechanism, we will not worry about the consistency of the theory of massive gravitons. In recent works, [12], [13], it was shown that one can avoid the singularity associated with the zero mass limit of the massive spin-2 field by giving mass to the graviton through spontaneous symmetry breaking, and to perform the physical analysis in the non-unitary gauge where the Higgs fields propagate.

The restriction that a metric on the manifold is not used, highly restricts the possible terms that could be written for the Lagrangian. This also has the added advantage that the metric of space-time is obtained dynamically after the symmetry is broken, and is found to be neutral under the $SU(N)\times SU(N)$ symmetry. All other massive gravitons, save for the combination $\left(e_\mu^a-f_\mu^a\right)$ transform under $SU(N)\times SU(N)$ and are therefore colored. It is straightforward to couple this model to matter, making use of vierbeins, Higgs fields and covariant derivatives. Much work is still needed to test the consistency of this theory at higher orders in perturbation. It will also be interesting to find out whether this construction can arise from other formulations such as noncommutative geometry, since the basic variables here whether vierbeins, Higgs fields or gauge fields are all matrix-valued, and therefore noncommuting.

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7 Appendix

We express the different parts of the action in component form. We shall only compute traces of Dirac matrices but not of $U(N)$ matrices. These will be calculated in the body of the paper, but only for the quadratic part of the Lagrangian. First we evaluate the non-Higgs terms

$$\frac{1}{2N}\int\limits_M Tr\left(i\left(\alpha+\beta\gamma_5\right)LL^{'}F+i\left(\overline{\alpha}+\overline{\beta}\gamma_5\right)L^{'}L\overline{F}+\left(i\lambda+\gamma_5\eta\right)LL^{'}LL^{'}\right),$$

which give

$$\begin{aligned}\frac{1}{N}\int\limits_M d^4x\epsilon^{\mu\nu\kappa\lambda}tr\Big(&\alpha_1^{'}i\left[E_\mu^a,E_\nu^{'a}\right]a_{\kappa\lambda}^1-\alpha_2^{'}\left\{E_\mu^a,E_\nu^{'a}\right\}a_{\kappa\lambda}^2\\
&+\beta_1^{'}i\left[F_\mu^a,F_\nu^{'a}\right]b_{\kappa\lambda}^1-\beta_2^{'}\left\{F_\mu^a,F_\nu^{'a}\right\}b_{\kappa\lambda}^2\\
&+\frac{1}{2}\left(\alpha_1+\beta_2\right)\left(i\left[E_\mu^a,E_\nu^{'b}\right]B_{\kappa\lambda ab}-\frac{1}{2}\epsilon_{abcd}\left\{E_\mu^a,E_\nu^{'b}\right\}B_{\kappa\lambda}^{cd}\right)\\
&+\frac{1}{2}\left(\alpha_1-\beta_2\right)\left(i\left[F_\mu^a,F_\nu^{'b}\right]C_{\kappa\lambda ab}+\frac{1}{2}\epsilon_{abcd}\left\{F_\mu^a,F_\nu^{'b}\right\}C_{\kappa\lambda}^{cd}\right)\\
&+\frac{1}{2}\left(-\alpha_2+\beta_1\right)\left(\left\{E_\mu^a,E_\nu^{'b}\right\}B_{\kappa\lambda ab}+\frac{1}{2}\epsilon_{abcd}i\left[E_\mu^a,E_\nu^{'b}\right]B_{\kappa\lambda}^{cd}\right)\\
&-\frac{1}{2}\left(\alpha_2+\beta_1\right)\left(\left\{F_\mu^a,F_\nu^{'b}\right\}C_{\kappa\lambda ab}-\frac{1}{2}\epsilon_{abcd}i\left[F_\mu^a,F_\nu^{'b}\right]C_{\kappa\lambda}^{cd}\right)\\
&+i\left(\lambda-\eta\right)\left(E_\mu^aE_\nu^{'a}E_\kappa^bE_\lambda^{'b}+E_\mu^aE_\nu^{'b}E_\kappa^bE_\lambda^{'a}-E_\mu^aE_\nu^{'b}E_\kappa^aE_\lambda^{'b}\right)\\
&+i\left(\lambda+\eta\right)\left(F_\mu^aF_\nu^{'a}F_\kappa^bF_\lambda^{'b}+F_\mu^aF_\nu^{'b}F_\kappa^bF_\lambda^{'a}-F_\mu^aF_\nu^{'b}F_\kappa^aF_\lambda^{'b}\right)\\
&+\epsilon_{abcd}\left(\left(\lambda-\eta\right)E_\mu^aE_\nu^{'b}E_\kappa^cE_\lambda^{'d}+\left(\lambda+\eta\right)F_\mu^aF_\nu^{'b}F_\kappa^cF_\lambda^{'d}\right)\Big),\end{aligned}$$

where

$$\begin{aligned}
\alpha_1^{'} &= (\alpha_1 + \beta_2 - \alpha_2 + \beta_1)\,, \\
\alpha_2^{'} &= (\alpha_1 + \beta_2 + \alpha_2 - \beta_1)\,, \\
\beta_1^{'} &= (\alpha_1 - \beta_2 - \alpha_2 - \beta_1)\,, \\
\beta_2^{'} &= (\alpha_1 - \beta_2 + \alpha_2 + \beta_1)\,.
\end{aligned}$$

The mixing terms

$$\frac{1}{2N}\int\limits_M Tr\left((i\tau + \gamma_5\xi)\,LH^{'}LH^{'}LL^{'} + (i\rho + \gamma_5\chi)\,HL^{'}HL^{'}LL^{'}\right),$$

give the contributions

$$\begin{aligned}
&\frac{1}{N}\int\limits_M d^4x\epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}tr\left((\tau - \xi)\,E^a_\mu H^{'}F^b_\nu H^{'}E^c_\kappa E^{'d}_\lambda - (\tau + \xi)\,F^a_\mu H^{'}E^b_\nu H^{'}F^c_\kappa F^{'d}_\lambda\right. \\
&\qquad\qquad\left. + (\rho - \chi)\,HF^{'a}_\mu HE^{'b}_\nu E^c_\kappa E^{'d}_\lambda + (\rho + \chi)\,HE^{'a}_\mu HF^{'b}_\nu F^c_\kappa F^{'d}_\lambda\right) \\
&+ \frac{1}{N}\int\limits_M d^4x\epsilon^{\mu\nu\kappa\lambda}\left(\eta_{ab}\eta_{cd} - \eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc}\right)tr\left((\tau - \xi)\,E^a_\mu H^{'}F^b_\nu H^{'}E^c_\kappa E^{'d}_\lambda\right. \\
&\qquad\qquad + (\tau + \xi)\,F^a_\mu H^{'}E^b_\nu H^{'}F^c_\kappa F^{'d}_\lambda \\
&\qquad\qquad + (\rho - \chi)\,HF^{'a}_\mu HE^{'b}_\nu E^c_\kappa E^{'d}_\lambda \\
&\qquad\qquad \left. + (\rho + \chi)\,HE^{'a}_\mu HF^{'b}_\nu F^c_\kappa F^{'d}_\lambda\right).
\end{aligned}$$

References

[1] H. Weyl, *Space, Time, Matter,* (Dover, New York, 1952).

[2] E. Cartan, *The Theory of Spinors* (Dover, New York 1981).

[3] R. Utiyama, *Phys. Rev.* **101,** 1597 (1956) .

[4] T. W. B. Kibble, *J. Math. Phys.* **2,** 212 (1961).

[5] S. R. Coleman and J. Mandula, *Phys. Rev.* **159,** 1251 (1967).

[6] C. J. Isham, Abdus Salam and J. Strathdee, *Phys. Rev.* D **8,** 2600 (1973).

[7] I. Avramidi, *Phys. Lett.* B **576,** 195 (2003); *Class. Quant. Grav.* **21,** 103 (2004).

[8] I. Avramidi, *Gauged Gravity via Spectral Asymptotics of Non-Laplace Type Operators,* `hep-th/0406026`.

[9] H. van Dam, M. Veltman, *Nucl. Phys.* B **22,** 397 (1970).

[10] D. G. Boulware, S. Deser, *Phys. Rev.* D **6,** 3368 (1972).

[11] T. Damour and I. I. Kogan, *Phys. Rev.* D **66,** 104024 (2002).

[12] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, *Ann. Phys.* **305,** 96 (2003).

[13] A. H. Chamseddine, *Phys. Lett.* B **557,** 247 (2003).

[14] R. Delbourgo, M. A. Rashid, Abdus Salam and J. Strathdee, in *Seminar on High-Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, Austria, 1965).

[15] A. H. Chamseddine, *Phys. Rev.* D **69,** 024015 (2004).

[16] M. Fierz, W. Pauli, *Proc. R. Soc. London, Ser.* A **173,** 211 (1939).

[17] A. H. Chamseddine, Abdus Salam and J. Strathdee, *Nucl. Phys.* B **136,** 248 (1978).

[18] S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **177,** 2239 (1969); C. Callan, S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **177,** 2247 (1969).

[19] T. Damour, *Experimental Tests of Gravitational Theory,* in Particle Data Group, S. Eidelman et al, *Phys. Lett.* B **592** 1 (2004).